Opt Quant Electron (2017) 49:285
DOI 10.1007/s11082-017-1127-z

Novel archetypes of new coupled Konno–Oono equation by using sine–Gordon expansion method

Gülnur Yel[1] · Haci Mehmet Baskonus[2] · Hasan Bulut[3]

Received: 17 April 2017 / Accepted: 1 August 2017
© Springer Science+Business Media, LLC 2017

Abstract In this manuscript, we investigate the analytical solutions of the new coupled Konno–Oono equation by using the sine–Gordon expansion method. We successfully obtained some new soliton solutions to this model. We plot the 2D and 3D surfaces of all the found soliton solutions in this study. The Wolfram Mathematica 9 is used in performing all the computations in this study.

Keywords The sine–Gordon expansion method · The new coupled Konno–Oono equation · Complex hyperbolic solutions

1 Introduction

Nowadays nonlinear partial differential equations (NLPDEs) are equations that describes various nonlinear physical aspects arising in various fields of nonlinear sciences such as chemistry, physics, quantum mechanics, fluid mechanics, electricity, ecology, meteorology etc.

For investigation of the analytical solutions to the different types of NLPDEs, various analytical approaches have been developed such as the new generalized (G'/G)-expansion method (Alam and Belgacem 2016), the improved (G'/G)-expansion method (Zhang et al.

✉ Haci Mehmet Baskonus
hmbaskonus@gmail.com

Gülnur Yel
gulnuryel33@gmail.com

Hasan Bulut
hbulut@firat.edu.tr

1 Department of Mathematics Education, Final International University, Kyrenia, Cyprus

2 Department of Computer Engineering, Munzur University, Tunceli, Turkey

3 Department of Mathematics, University of Firat, Elazig, Turkey

Published online: 07 August 2017

2010; Seadawy et al. 2017), exp($-\Omega(\xi)$)-expansion function method (Seadawy et al. 2017; Khater 2016), modified exp($-\Omega(\xi)$)-expansion function method (Kocak et al. 2016), modified simple equation method (Lu et al. 2017), the improved Bernoulli sub-equation function method (Baskonus and Bulut 2015), the Adomian decomposition method (Adomian 1990; Adomian and Rach 1990), the sine–cosine method (Wazwaz 2004), the generalized Kudryashov method (Islam et al. 2015), modified trial equation method (Bulut et al. 2017; Kocak et al. 2014), generalized the extended tanh-function method (Xue-Dong et al. 2003), the modified extended tanh-function method (Khater 2016; Zahran and Khater 2016), homotopy analysis method (Liao 2005), homotopy perturbation method (Bulut and Baskonus 2010) and so on.

In (1994), Konno and Oono presented more general version of coupled integrable dispersionless system defined as

$$
\begin{aligned}
&q_{xt} - 2\alpha q r_x - 2\beta q s_x + \gamma (rs)_x = 0,\\
&r_{xt} - 2\alpha r r_x + 2\beta (2q q_x + r_x s) - 2\gamma q_x r = 0,\\
&s_{xt} - 2\beta s s_x + 2\alpha (2q q_x + r s_x) - 2\gamma s q_x = 0,
\end{aligned}
\tag{1.1a}
$$

where α, β, γ are constants. This system physically defines a current-fed string interacting with an external magnetic field in three-dimensional Euclidean space (Konno and Oono 1994; Konno and Kakuhata 1995, 1996; Souleymanou et al. 2012). It also appears geometrically as the parallel transport of each point of the curve along the direction of time where the connection is magnetic-valued (Konno and Oono 1994; Konno and Kakuhata 1995, 1996; Souleymanou et al. 2012; Khalique 2012).

When special values of some coefficients is taken in Eq. (1.1a), this system converted into new Konno–Oono equation system which is a coupled integrable dispersionless equations given as following (Konno and Oono 1994).

$$
\begin{aligned}
&v_t(x,t) + 2u(x,t)u_x(x,t) = 0,\\
&u_{xt}(x,t) - 2v(x,t)u(x,t) = 0.
\end{aligned}
\tag{1.1b}
$$

This new Konno–Oono equation system attract attention some scientist from all over the world. They have investigated this model in terms of new and different properties by using some mathematical approaches. In Bashar et al. (2016), the tanh-function method and extended tanh-function method have been obtained to construct the exact soliton solutions for Eq. (1.1b). In Khan and Akbar (2013), the modified simple equation method has been employed for getting some properties of Eq. (1.1b) and kink solutions, bell-shaped solutions were obtained by this method.

In this study, we aim of submit for soliton structures of Eq. (1.1b) by using the sine–Gordon expansion method. This paper has been organised as follows: In Sect. 2, we present the description the method. In Sect. 3, SGEM has been applied to the new coupled Konno–Oono equation system. Finally, in Sect. 4, we presented the conclusions of the study comprehensively.

2 Fundamental properties of the method

In this section, we describe the sine–Gordon expansion method. Before giving the general features of the sine–Gordon expansion method, we need to explain how we get two significant equations. Firstly, lets suppose that the sine–Gordon equation is given as following (Yan 1996; Yan and Zhang 1999; Zhen-Ya et al. 1999; Baskonus et al. 2017):

$$u_{xx} - u_{tt} = m^2 \sin(u), \tag{2.1}$$

where $u = u(x, t)$, m is a real constant. Applying the wave transform $u = u(x, t) = U(\xi)$, $\xi = \mu(x - ct)$ to Eq. (2.1),

$$\begin{aligned} u_x &= \frac{dU}{d\xi} \cdot \frac{d\xi}{dx} = \mu \cdot U', \quad u_{xx} = \frac{d(u_x)}{d\xi} \cdot \frac{d\xi}{dx} = \mu^2 U'', \\ u_t &= \frac{dU}{d\xi} \cdot \frac{d\xi}{dt} = -\mu \cdot c \cdot U', \quad u_{tt} = \frac{d(u_t)}{d\xi} \cdot \frac{d\xi}{dt} = c^2 \mu^2 U'', \end{aligned} \tag{2.2}$$

Equation (2.2) are obtained. After putting Eq. (2.2) into Eq. (2.1) and simplification, we get the following nonlinear ordinary differential equation:

$$U'' = \frac{m^2}{\mu^2(1 - c^2)} \sin(U), \tag{2.3}$$

where $U = U(\xi)$, ξ is the amplitude of the travelling wave and c is the velocity of the travelling wave. We can full simplificate Eq. (2.3) and it can be written as follows:

$$\left[\left(\frac{U}{2}\right)'\right]^2 = \frac{m^2}{\mu^2(1 - c^2)} \sin^2\left(\frac{U}{2}\right) + K, \tag{2.4}$$

where K is the constant of integration. Substituting $K = 0, w(\xi) = \frac{U}{2}$ and $a^2 = \frac{m^2}{\mu^2(1-c^2)}$ in Eq. (2.4), gives:

$$w' = a \sin(w), \tag{2.5}$$

setting $a = 1$ in Eq. (2.5), gives:

$$w' = \sin(w). \tag{2.6}$$

Solving Eq. (2.6) by variables separable, we obtain the two significant properties that are given as follows:

$$\sin(w) = \sin(w(\xi)) = \left.\frac{2pe^{\xi}}{p^2 e^{2\xi} + 1}\right|_{p=1} = \operatorname{sech}(\xi), \tag{2.7}$$

$$\cos(w) = \cos(w(\xi)) = \left.\frac{p^2 e^{2\xi} - 1}{p^2 e^{2\xi} + 1}\right|_{p=1} = \tanh(\xi), \tag{2.8}$$

where p is the integral constant and non-zero.

After these two important properties, when it comes to the description of sine–Gordon expansion method, to obtain the solution of the nonlinear partial differential equation of the following form:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, u_{xxt}, \ldots), \tag{2.9}$$

we consider,

$$U(\xi) = \sum_{i=1}^{n} \tanh^{i-1}(\xi)[B_i \operatorname{sech}(\xi) + A_i \tanh(\xi)] + A_0. \tag{2.10}$$

Equation (2.10) can be rearranged according to Eqs. (2.7) and (2.8) as follows:

$$U(w) = \sum_{i=1}^{n} \cos^{i-1}(w)[B_i \sin(w) + A_i \cos(w)] + A_0. \tag{2.11}$$

We apply the balance principle to determine the value of n under the highest power nonlinear and highest derivative in the ordinary differential equation. We suppose that the summation of coefficients of $\sin^i(w)\cos^j(w)$ with the same power is zero, this yields a system of equations. With aid of the Wolfram Mathematica 9, we solve the system of equations to obtain the values of A_i, B_i, μ and c. At the end of the method, substituting the values of A_i, B_i, μ and c into Eq. (2.10), we get the new travelling wave solutions to Eq. (2.9).

3 Application

In this section, we implement the SGEM to the new coupled Konno–Oono equation system. Lets start supposing that the travelling wave transformation as following:

$$\begin{aligned} u(x,t) &= U(\xi), \quad \xi = \mu(x - ct), \\ v(x,t) &= V(\xi), \quad \xi = \mu(x - ct), \end{aligned} \tag{3.1}$$

where μ is the wave height and c is the wave velocity. We obtained the following partial derivatives of $U(\xi)$ and $V(\xi)$ functions respect to ξ:

$$u_x = \frac{dU}{d\xi} \cdot \frac{d\xi}{dx} = \mu \cdot U', \quad u_{xt} = \frac{d(\mu \cdot U')}{d\xi} . \frac{d\xi}{dt} = -c\mu^2 U'', \quad v_t = \frac{dV}{d\xi} . \frac{d\xi}{dt} = -\mu c V', \tag{3.2}$$

Putting Eq. (3.2) into Eq. (1.1b), we get the following equations

$$-c\mu^2 U'' - 2UV = 0, \tag{3.3}$$

$$-c\mu V' + 2\mu U U' = 0. \tag{3.4}$$

Thus, integrating Eq. (3.4) with respect to ξ, we get:

$$V = \frac{1}{c}\left(U^2 + p\right), \tag{3.5}$$

where p is the integration constant. Putting Eq. (3.5) into Eq. (3.3), we obtain:

$$c^2\mu^2 U'' + 2U^3 + 2pU = 0, \tag{3.6}$$

having c, μ are both real constants and non-zero. When we reconsider the Eq. (2.11) for homogenous balance method between U'' and U^3, the value of n is obtained as following:

$$n = 1. \tag{3.7}$$

If we put $n = 1$ into Eq. (2.11), we get the followings equations:

$$U(w) = B_1 \sin(w) + A_1 \cos(w) + A_0, \tag{3.8}$$

$$U'(w) = B_1 \cos(w)\sin(w) - A_1 \sin^2(w), \tag{3.9}$$

$$U''(w) = B_1\left[\cos^2(w)\sin(w) - \sin^3(w)\right] - 2A_1 \sin^2(w)\cos(w). \tag{3.10}$$

Putting Eqs. (3.8) and Eq. (3.10) into Eq. (3.6), we get the following trigonometric functions equation:

$$\begin{aligned} &2pA_0 + 2A_0^3 + 2pA_1\cos[w] - 2c^2\mu^2 A_1 \cos[w]\sin^2[w] \\ &\quad + 6A_0^2A_1\cos[w] + 6A_0A_1^2\cos^2[w] + 2A_1^3\cos^3[w] + 2pB_1\sin[w] \\ &\quad + c^2\mu^2B_1\cos^2[w]\sin[w] - c^2\mu^2B_1\sin^3[w] + 6A_0^2B_1\sin[w] \\ &\quad + 12A_0A_1B_1\cos[w]\sin[w] + 6A_1^2B_1\cos^2[w]\sin[w] + 6A_0B_1^2\sin^2[w] \\ &\quad + 6A_1B_1^2\cos[w]\sin^2[w] + 2B_1^3\sin^3[w] = 0. \end{aligned} \tag{3.11}$$

Setting each summation of the coefficients of the trigonometric identities of the same power to be zero, yields the following algebraic set of equations:

$$\begin{aligned} &\text{Constants: } 2pA_0 + 2A_0^3 + 6A_0B_1^2, \\ &\cos[w]: \ 2pA_1 + 6A_0^2A_1 + 2A_1^3, \\ &\cos[w]\sin^2[w]: \ -2c^2\mu^2A_1 + 6A_1B_1^2 - 2A_1^3, \\ &\cos^2[w]: \ 6A_0A_1^2 - 6A_0B_1^2, \\ &\sin[w]: \ 2pB_1 + 6A_0^2B_1 - c^2\mu^2B_1 + 2B_1^3, \\ &\cos^2[w]\sin[w]: \ 2c^2\mu^2B_1 + 6A_1^2B_1 - 2B_1^3, \\ &\cos[w]\sin[w]: \ 12A_0A_1B_1. \end{aligned} \tag{3.12}$$

By solving the algebraic set of equations, we obtain following values of the coefficients:

$$\textbf{Case 1}: A_0 = 0, \quad A_1 = i, \quad B_1 = p = 1, \quad c = -\frac{2}{\mu}. \tag{3.13}$$

$$\textbf{Case 2}: A_0 = 0, \quad A_1 = -i, \quad B_1 = p = 1, \quad c = \frac{2}{\mu}. \tag{3.14}$$

$$\textbf{Case 3}: A_0 = 0, \quad A_1 = i, \quad B_1 = p = 1, \quad c = \frac{2}{\mu}. \tag{3.15}$$

If we use coefficients of Eqs. (3.13) into Eq. (3.8) along with Eq. (3.1) for $u(x, t)$ and into Eq. (3.5) for $v(x, t)$, we get the following complex hyperbolic function solutions for the Eq. (1.1b):

$$\begin{aligned} &u_1(x,t) = \mathrm{Sech}[(2t + x\mu)] + i\mathrm{Tanh}[(2t + x\mu)], \\ &v_1(x,t) = -\frac{1}{2}\mu\Big(1 + (\mathrm{Sech}[(2t + x\mu)] + i\mathrm{Tanh}[(2t + x\mu)])^2\Big), \end{aligned} \tag{3.16}$$

where μ is a real constant and non-zero. If we plot two- and three-dimensional surfaces of Eq. (3.16) for suitable values of parameters, we get following surfaces.

If we use coefficients of Eq. (3.14) into Eq. (3.8) along with Eq. (3.1) for $u(x, t)$ and Eq. (3.5) for $v(x, t)$, we get another complex hyperbolic function solution for the Eq. (1.1b):

$$u_2(x,t) = \mathrm{Sech}[(-2t + x\mu)] - i\mathrm{Tanh}[(-2t + x\mu)],$$
$$v_2(x,t) = \frac{1}{2}\mu\left(1 + (\mathrm{Sech}[(-2t + x\mu)] - i\mathrm{Tanh}[(-2t + x\mu)])^2\right), \tag{3.17}$$

where μ is a real constant and non-zero. If we plot two- and three-dimensional surfaces of Eq. (3.17) for suitable values of parameters, we get following surfaces.

Using coefficients of Eq. (3.15) into Eq. (3.8) along with Eq. (3.1) for $u(x, t)$ and Eq. (3.5) for $v(x, t)$, we get another complex hyperbolic function solution to the Eq. (1.1b):

$$u_3(x,t) = \mathrm{Sech}[(-2t + x\mu)] + i\mathrm{Tanh}[(-2t + x\mu)],$$
$$v_3(x,t) = \frac{1}{2}\mu\left(1 + (\mathrm{Sech}[(-2t + x\mu)] + i\mathrm{Tanh}[(-2t + x\mu)])^2\right), \tag{3.18}$$

where μ is a real constant and non-zero. If we plot two- and three-dimensional surfaces of Eq. (3.18) for suitable values of parameters, we get following surfaces.

4 Physical meanings, discussions and conclusions

In this study, with the aid of symbolic software program, we applied the sine–Gordon expansion method in investigating the solutions of the new coupled Konno–Oono equation. We successfully constructed some new complex hyperbolic function solutions that haven't been submitted to literature beforehand. We observe that these new solutions include some significant various the type of solutions such as kink-type solution and singular soliton

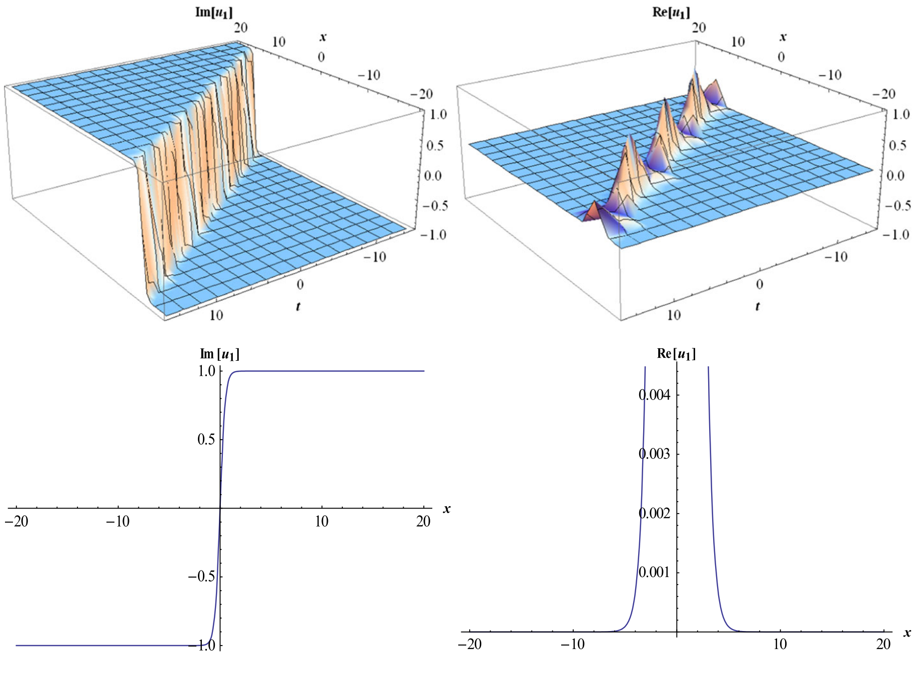

Fig. 1 The 2D and 3D surfaces of Eq. (3.16) for u_1 with $\mu = 2$, where $-20 < x < 20$, $-15 < t < 15$ and $t = 0.002$ for 2D surfaces

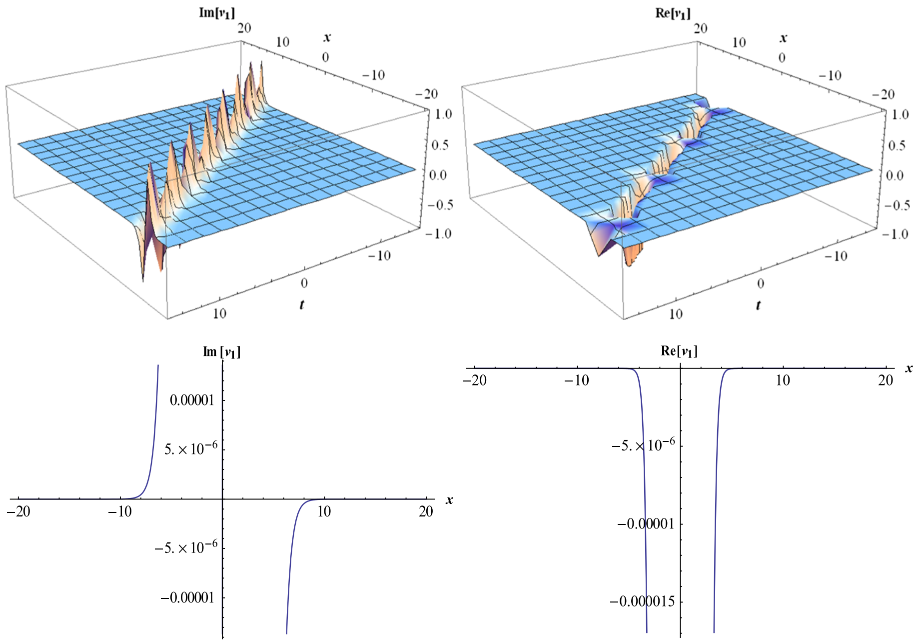

Fig. 2 The 2D and 3D surfaces of Eq. (3.16) for v_1 with $\mu = 2$, where $-20 < x < 20$, $-15 < t < 15$ and $t = 0.002$ for 2D surfaces

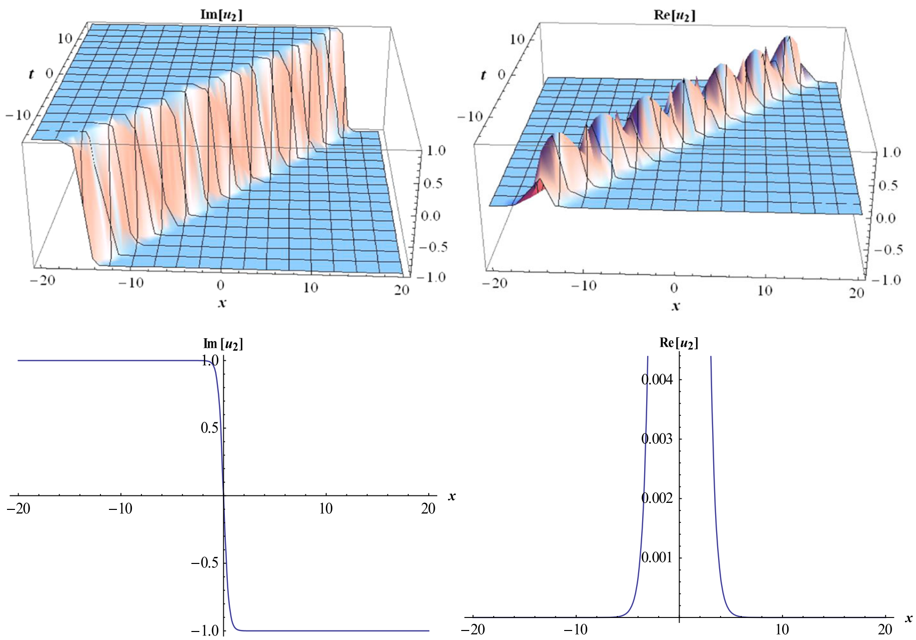

Fig. 3 The 2D and 3D surfaces of Eq. (3.17) for u_2 with $\mu = 2$, where $-20 < x < 20$, $-15 < t < 15$ and $t = 0.002$ for 2D surfaces

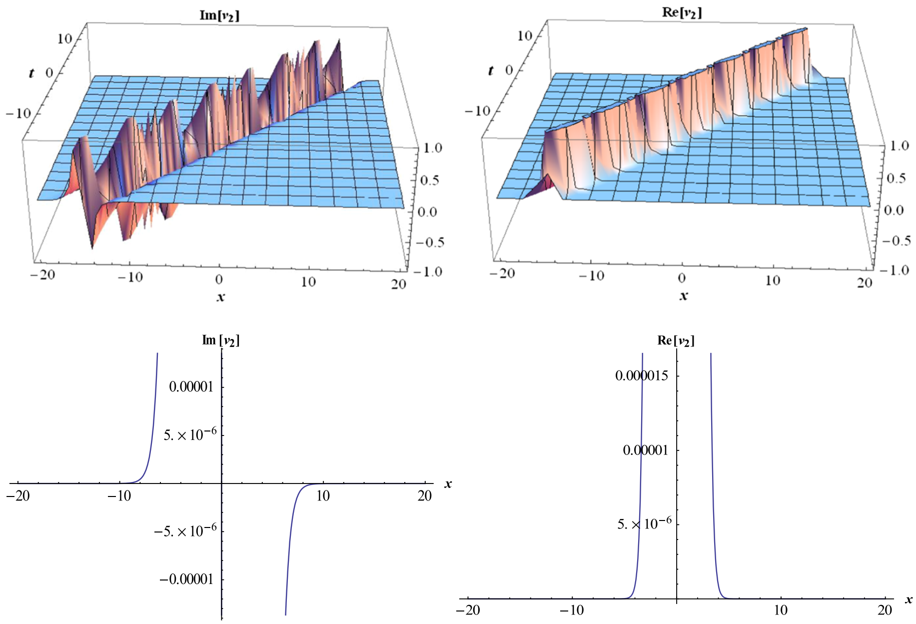

Fig. 4 The 2D and 3D surfaces of Eq. (3.17) for v_2 with $\mu = 2$, where $-20 < x < 20$, $-15 < t < 15$ and $t = 0.002$ for 2D surfaces

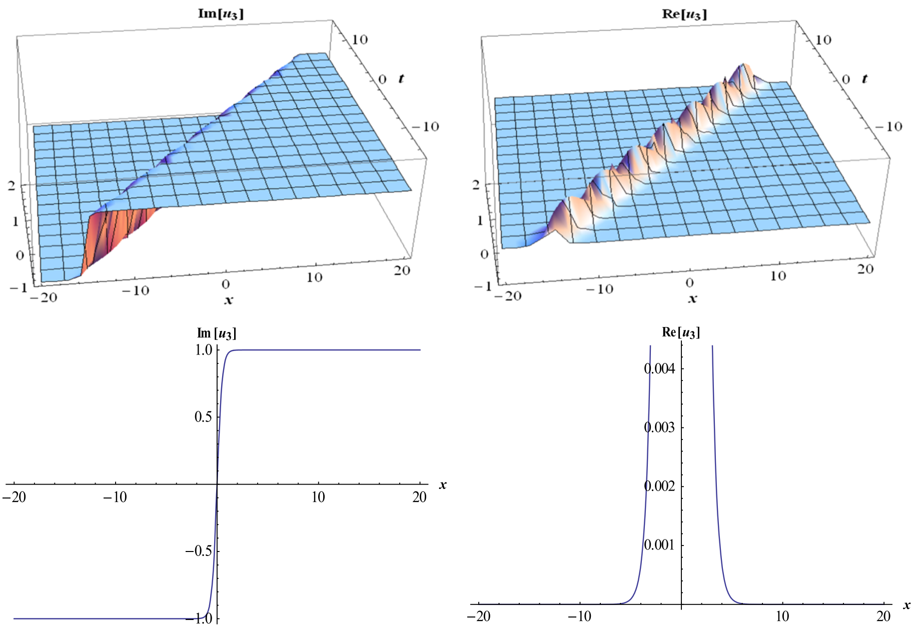

Fig. 5 The 2D and 3D surfaces of Eq. (3.18) for u_3 with $\mu = 2$, where $-20 < x < 20$, $-15 < t < 15$ and $t = 0.002$ for 2D surfaces

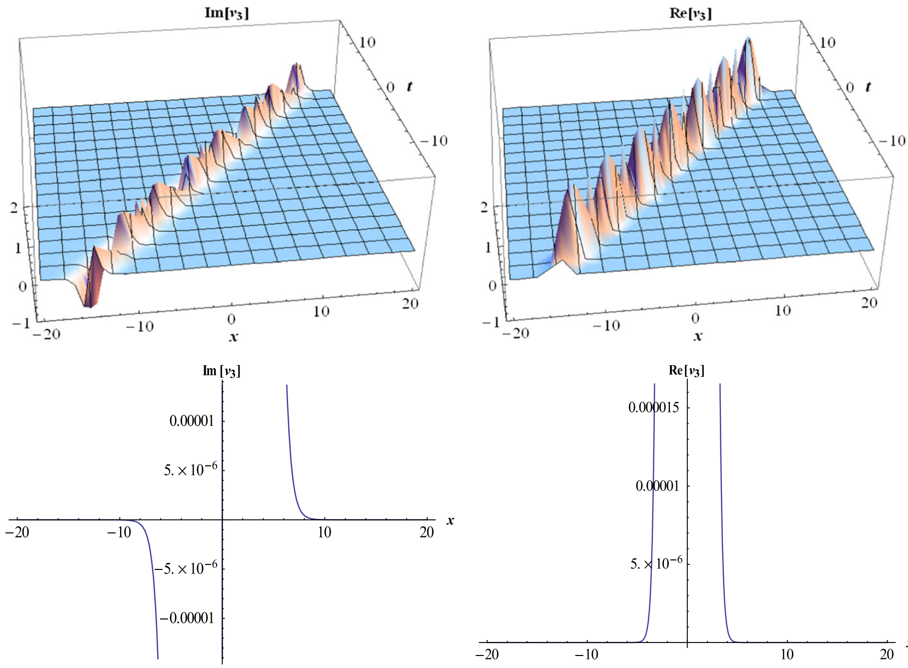

Fig. 6 The 2D and 3D surfaces of Eq. (3.18) for v_3 with $\mu = 2$, where $-20 < x < 20$, $-15 < t < 15$ and $t = 0.002$ for 2D surfaces

solutions. When we look at the Figs. 1, 3 and 5 three-dimensional surfaces of u_1, u_2 and u_3 are kink-type and singular soliton surfaces. For Figs. 2, 4 and 6 three-dimensional surfaces of v_1, v_2 and v_3 are singular soliton surfaces.

Moreover, such hyperbolic functions are of different physical meanings as well. We observed that the obtained solutions in this study have some important physical meanings, for instance; tangent hyperbolic function arises in the calculation of magnetic moment and rapidity of special relativity, the hyperbolic secant arises in the profile of a laminar jet (Weisstein 2002).

It is estimated that the newly obtained these hyperbolic function solutions in this paper may help to explain some complex physical aspects in the nonlinear physical sciences and are related to such physical properties. To the best of our knowledge, the application of the sine–Gordon expansion method has not been submitted to the literature in advance.

References

Adomian, G.: A review of the decomposition method and some recent results for nonlinear equations. Math. Comput. Model. **13**(7), 17–43 (1990)

Adomian, G., Rach, R.: Equality of partial solutions in the decomposition method for linear or nonlinear partial differential equations. Comput. Math Appl. **19**(12), 9–12 (1990)

Alam, N., Belgacem, F.B.M.: New generalized (G'/G) expansion method applications to coupled Konno–Oono equation. Adv. Pure Math. **6**, 168–179 (2016)

Bashar, A., Mondal, G., Khan, K., Bekir, A.: Traveling wave solutions of new coupled Konno–Oono equation. New Trends Math. Sci. **4**(2), 296–303 (2016)

Baskonus, H.M., Bulut, H.: On the complex structures of Kundu–Eckhaus equation via improved Bernoulli sub-equation function method. Waves Random Complex Media **25**(4), 720–728 (2015)

Baskonus, H.M., Sulaiman, T.A., Bulut, H.: On the novel wave behaviors to the coupled nonlinear Maccari's system with complex structure. Opt. Int. J. Light Electron Opt. **131**, 1036–1043 (2017)

Bulut, H., Baskonus, H.M.: A study on the numerical solution of the third-order dispersive equations with homotopy perturbation method. e-J. New World Sci. Acad. **5**(1), 18 (2010)

Bulut, H., Yel, G., Baskonus, H.M.: Novel structure to the coupled nonlinear Maccari's system by using modified trial equation. Adv. Math. Models Appl. **2**(1), 14–19 (2017)

Islam, M.S., Khan, K., Arnous, A.H.: Generalized Kudryashov method for solving some (3 + 1)-dimensional nonlinear evolution equations. New Trends Math. Sci. **3**(3), 46–57 (2015)

Khalique, C.M.: Exact solutions and conservation laws of a coupled integrable dispersionless system. Filomat **26**(5), 957–964 (2012)

Khan, K., Akbar, M.A.: Traveling Wave Solutions of Some Coupled Nonlinear Evolution Equations. Hindawi Publishing Corporation, Cairo (2013)

Khater, M.M.A.: Exact traveling wave solutions for an important mathematical physics model. J. Appl. Math. Bioinform. **6**(1), 37–48 (2016a)

Khater, M.M.A.: Exact traveling wave solutions for the generalized Hirota–Satsuma couple KdV system using the exp $(-\varphi(\xi))$-expansion method. Cogent Math. **3**(1), 1–16 (2016b)

Kocak, Z.F., Bulut, H., Yel, G.: The solution of fractional wave equation by using modified trial equation method and homotopy analysis method. AIP Conf. Proc. **1637**, 504–512 (2014)

Kocak, Z.F., Bulut, H., Koc, D.A., Baskonus, H.M.: Prototype traveling wave solutions of new coupled Konno–Oono equation. Opt. Int. J. Light Electron Opt. **127**(22), 10786–10794 (2016)

Konno, K., Kakuhata, H.: Interaction among growing, decaying and stationary solitons for coupled integrable dispersionless equations. J. Phys. Soc. Jpn. **64**, 2707–2709 (1995)

Konno, K., Kakuhata, H.: Novel solitonic evolutions in a coupled integrable, dispersionless system. J. Phys. Soc. Jpn. **65**, 713–721 (1996)

Konno, K., Oono, H.: New coupled integrable dispersionless equations. J. Phys. Soc. Jpn. **63**, 377–378 (1994)

Liao, S.: Comparison between the homotopy analysis method and homotopy perturbation method. Appl. Math. Comput. **169**(2), 1186–1194 (2005)

Lu, D., Seadawy, A.R., Khater, M.M.A.: Bifurcations of new multi soliton solutions of the van der Waals normal form for fluidized granular matter via six different methods. Results Phys. **7**, 2028–2035 (2017)

Seadawy, A.R., Lu, D., Khater, M.M.A.: Bifurcations of solitary wave solutions for the three dimensional Zakharov–Kuznetsov–Burgers equation and Boussinesq equation with dual dispersion. Opt. Int. J. Light Electron Opt. **43**, 104–114 (2017a)

Seadawy, A.R., Lu, D., Khater, M.M.A.: Solitary wave solutions for the generalized Zakharov–Kuznetsov–Benjamin–Bona–Mahony nonlinear evolution equation. J. Ocean Eng. Sci. **2**(2), 137–142 (2017b)

Souleymanou, A., Kuetche, V.K., Bouetou, T.B., Kofane, T.C.: Traveling wave-guide channels of a new coupled integrable dispersionless system. Commun. Theor. Phys. **57**, 10–14 (2012)

Wazwaz, A.M.: A sine–cosine method for handling nonlinear wave equations. Math. Comput. Model. **40**, 499–508 (2004)

Weisstein, E.W.: Concise Encyclopedia of Mathematics, 2nd edn. CRC Press, New York (2002)

Xue-Dong, Z., Yong, C., Biao, L., Hong-Qing, Z.: A new generalization of extended tanh-function method for solving nonlinear evolution equations. Commun. Theor. Phys. **39**(6), 647 (2003)

Yan, C.: A simple transformation for nonlinear waves. Phys. Lett. A **22**(4), 77–84 (1996)

Yan, Z., Zhang, H.: New explicit and exact travelling wave solutions for a system of variant Boussinesq equations in mathematical physics. Phys. Lett. A **252**, 291–296 (1999)

Zahran, E.H.M., Khater, M.M.A.: The modified extended tanh function method and its applications to the Bogoyavlenskii equation. Appl. Math. Model. **40**, 1769–1775 (2016)

Zhang, J., Jiang, F., Zhao, X.: An improved (G'/G)-expansion method for solving nonlinear evolution equations. Int. J. Math. **87**, 1716–1725 (2010)

Zhen-Ya, Y., Hong-oing, Z., En-Gui, F.: New explicit and travelling wave solutions for a class of nonlinear evolution equations. Acta Phys. Sin. **48**(1), 1–5 (1999)